Relations and Orders in Discrete Mathematics

Properties, Equivalence, Partial Orders, Lattices, and Higman's Lemma - CS205M

Khushraj Madnani

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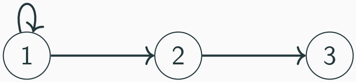
What is a Relation?

- A **binary relation** R on a set A is a subset of $A \times A$.
- Notation: aRb means $(a, b) \in R$.
- Example: On $A = \{1, 2, 3\}$, $R = \{(1, 1), (1, 2), (2, 3)\}$.



Reflexive Property

- **Definition**: $\forall a \in A, aRa$.
- Example: Equality ($=$) on $\mathbb{Z}$: $n = n$.
- Counterexample: Less than ($<$) is not reflexive.

Symmetric Property

- **Definition**: If *aRb*, then *bRa*.
- Example: "Is sibling of" (if A is sibling of B, vice versa).
- Counterexample: "Parent of" is not symmetric.

Transitive Property

- **Definition**: If aRb and bRc, then aRc.
- Example: "Ancestor of" (A to B, B to C $\implies$ A to C).
- Counterexample: "Friend of" may not be transitive.

Antisymmetric Property

- **Definition**: If aRb and bRa, then $a = b$.
- Example: $\leq$ on numbers (if $a \leq b$ and $b \leq a$, $a = b$).
- Counterexample: "Sibling" is not antisymmetric.

Implies $A = B$

Closure of Relations

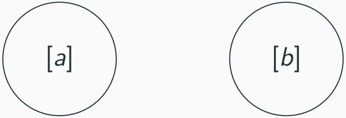
- **Definition**: Smallest relation containing R with property P.
- **Reflexive Closure**: Add (a, a) for all $a \in A$.
- Example: $R = \{(1, 2)\}$, reflexive closure adds $(1, 1), (2, 2)$.

Symmetric and Transitive Closures

- **Symmetric Closure**: Add (b, a) for each (a, b).
- **Transitive Closure**: Add all implied paths (R^*).
- Example: $R = \{(1, 2), (2, 3)\}$, transitive closure adds $(1, 3)$.

Equivalence Relations

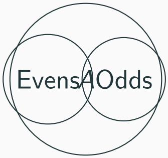
- **Definition**: Reflexive, symmetric, transitive.
- Example: Congruence mod n: $a \equiv b \pmod{5}$.
- Induces **equivalence classes**: $[a] = \{b \mid aRb\}$.



Disjoint classes

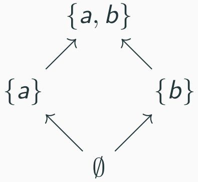
Partitions

- **Definition**: Disjoint, non-empty subsets whose union is A.
- Equivalence relation $\leftrightarrow$ partition via classes.
- Example: Integers by parity (evens, odds).



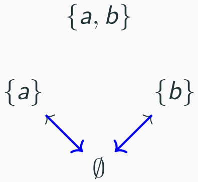
Partial Orders

- **Definition**: Reflexive, antisymmetric, transitive (poset).
- Example: $\subseteq$ on power set of $\{a, b\}$.
- **Hasse Diagram**: Shows order without transitive edges.



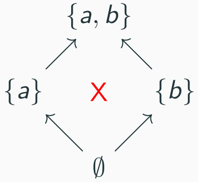
Greatest Lower Bound (GLB)

- **Definition**: Largest c s.t. $c \leq a$ and $c \leq b$ (meet, $a \wedge b$).
- Example: In power set, $\{a\} \wedge \{b\} = \emptyset$.
- In $\mathbb{Z}$ with $\leq$, $\text{GLB}(4, 6) = \min(4, 6)$.



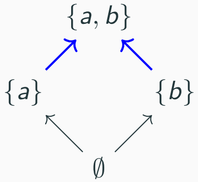
Incomparable Elements

- **Definition**: a, b where neither $a \leq b$ nor $b \leq a$.
- Example: $\{a\}$ and $\{b\}$ in power set.
- **Antichain**: Set of pairwise incomparable elements.



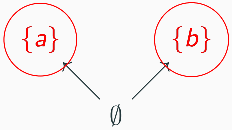
Least Upper Bound (LUB)

- **Definition**: Smallest c s.t. $a \leq c$ and $b \leq c$ (join, $a \vee b$).
- Example: In power set, $\{a\} \vee \{b\} = \{a, b\}$.
- May not exist (e.g., $\mathbb{Q}$ and $\{x \mid x^2 < 2\}$).



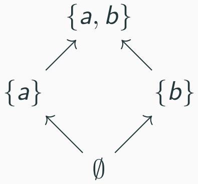
Maximal vs Maximum Elements

- **Maximal**: No element strictly greater.
- **Maximum**: $\geq$ all elements (unique).
- Example: Two tops in poset $\implies$ maximal, no maximum.



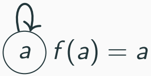
Lattices

- **Definition**: Poset where every pair has LUB and GLB.
- Example: Power set ($\vee = \cup$, $\wedge = \cap$).
- **Bounded**: Has top ($\top$) and bottom ($\bot$).



Fixed Points

- **Definition**: a where $f(a) = a$ for monotone f on poset.
- **Knaster-Tarski**: Monotone f on lattice has fixed points.
- Example: Program semantics (loop invariants).

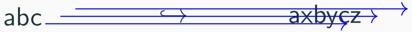


Well-Quasi-Ordering (WQO)

- **Definition**: Quasi-order (reflexive, transitive) with no infinite antichain or descending chain.
- Example: $\mathbb{N}$ with $\leq$ is well-ordered.
- Importance: Termination guarantees.

3
$\uparrow$
2No infinite $\downarrow$
$\uparrow$
1

- **Statement**: Σ^* (words over finite alphabet) is WQO under subsequence embedding.
- Subsequence: Letters of w appear in v in order.
- Example: "abc" embeds in "axbycz".



Multiset Ordering

- **Definition**: $M < N$ if M obtained by replacing elements in N with smaller ones.
- Example: $\{a, a, b\} > \{a, c\}$ if $c < b$, $a < b$.
- Terminates: No infinite descending chains.

$$\{a, a, b\} \qquad > \qquad \{a, c\}$$

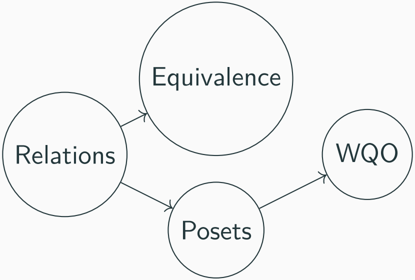
Proving Multiset Termination

- Map multisets to sorted words in Σ^* (WQO alphabet).
- Multiset reduction $\implies$ subsequence embedding.
- Higman's Lemma: No infinite descending chains.
- Application: Termination in term rewriting.

$$\{a, a, b\} \quad \mapsto \quad aab \quad > \quad ac$$

Summary

- Relations: Reflexive, symmetric, transitive, antisymmetric.
- Equivalence: Partitions via classes.
- Posets: GLB, LUB, maximal/maximum, lattices.
- WQO: No bad sequences; Higman's for words, multisets.



References

- Rosen, "Discrete Mathematics and Its Applications."
- Davey & Priestley, "Introduction to Lattices and Order."
- Higman, G. (1952). Ordering by divisibility in abstract algebras.
- https://en.wikipedia.org/wiki/Well-quasi-ordering